On Nonlinear Filters Involving Transformation of the Time Variable

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Abstract—**A new type of nonlinear filter, called the *E*-filter, is introduced that involves a transformation of the independent variable of the input function. It is shown how an *E*-filter can be designed to filter out superimposed "noise" on a signal, leaving the large peaks of the signal unattenuated. Unlike a low-pass linear filter, the low-pass *E*-filter is almost frequency independent and so does not affect the amplitudes of large sharp peaks of the signal. It is shown that the *E*-filter can be realized in real time and that a wide class of *E*-filters have a filtering action which is independent of the dc level of the input signal.**

I. Introduction

THERE has been an increased interest in nonlinear filtering over recent years. The literature on nonlinear filtering tends to fall into two areas. In the first area, nonlinear filters are derived mathematically from various optimum or suboptimum solutions of nonlinear estimation problems [1]–[3]. The other area involves a deterministic approach and is typified by the class of homomorphic filters examined by Oppenheim *et al.* [4]. The approach in this paper falls into the latter category.

The class of homomorphic filters considered by Oppenheim *et al.* are characterized by having the property that if $s_1(t)$ and $s_2(t)$ are two signals combined by some rule denoted by $\circ$, then the filter has the property

$$\phi[s_1(t) \circ s_2(t)] = \phi[s_1(t)] \circ \phi[s_2(t)] \tag{1}$$

where ϕ represents the transformation for the filter.

This is synonomous with the additive property for the special case of linear systems where the $\circ$ corresponds to addition. The generalization of the homogeneous property is stated as

$$\phi[C: s(t)] = C: \phi[s(t)] \tag{2}$$

where C is a scalar and the colon denotes the product of the input with the scalar.

A filter which satisfies both (1) and (2) is called a *homomorphic* filter. A homomorphic filter can be realized by a number of systems in cascade as shown in Fig. 1. The first system A_0 has the property that

$$A_0[s_1(t) \circ s_2(t)] = A_0[s_1(t)] + A_0[s_2(t)] \tag{3}$$

and

$$A_0[C: s(t)] = C: A_0[s(t)]. \tag{4}$$

The system L is linear and A_0^{-1} is the inverse of A_0; i.e.,

$$A_0^{-1}\{A_0[s(t)]\} = s(t). \tag{5}$$

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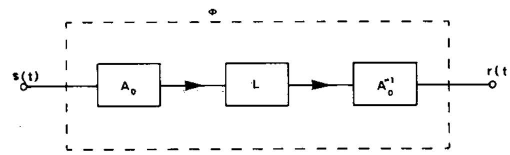

Fig. 1. Realization of homomorphic filter.

The application of this principle for convolved signals usually involves a process called cepstrum analysis and has been quite successful in the analysis and synthesis of speech [5]–[8].

In this paper we will consider a class of nonlinear filters of the same form as in Fig. 1, where the transformation A_0 involves a transformation of the independent variable t.

The transformation A_0 is defined by carrying out a transformation on the time variable t to a new variable denoted by e; i.e.,

$$A_0\{s(t)\} \triangleq f(e) \tag{6}$$

where

$$e = \theta[s(v),t], \qquad 0 \leq v \leq t < \infty. \tag{7}$$

The new variable e is a function of the time t and the history of the input function $s(\cdot)$ up until time t.

We also require that when $t = 0$ then $e = 0$; i.e.,

$$0 = \theta[s(0),0]. \tag{8}$$

The output $f(e)$ of A_0 is defined by

$$f(e) = s[\theta^{-1}(e)]. \tag{9}$$

The function $\theta(\cdot,\cdot)$ maps the input up to time t and the value of t onto e. For any given fixed input $s(t)$ the value of e will merely be a function of the single variable t. We will use the shorthand notation

$$e = \theta(t)$$

in this case where it is implicitly assumed that $\theta(t)$ is defined for *some particular input* $s(t)$.

We will assume that $\theta(t)$ is always monotonically increasing and possesses a first derivative. We will call any filter that has an A_0 defined by (6)–(9) and E-filter. The diagrammatic representation for an E-filter is shown in Fig. 2. In this paper we will show how the E-filter can be realized in real time and look at a particular example and its properties.

In the homomorphic filtering techniques developed by Oppenheim *et al.* [4], each succeeding stage in the operation is independent of the preceding stages. To prove that the whole transformation is invertible (and hence that the filter is homomorphic) one merely had to prove that each stage is invertible.

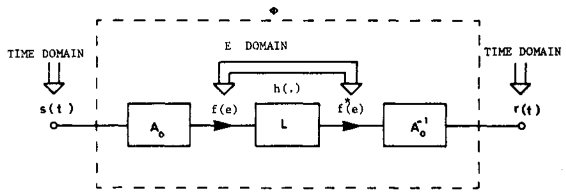

Fig. 2. Schematic diagram of E-filter.

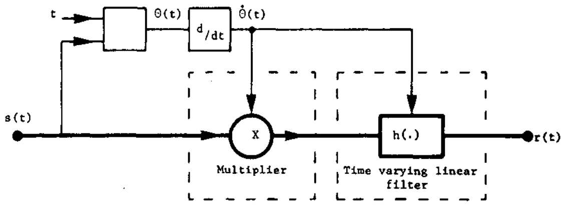

Fig. 3. Real time realization of the E-filter.

The situation is much more complicated in the case of the E-filter. Both the transformation A_0 and A_0^{-1} are dependent on the input signal $s(t)$ to the filter and so the same simple cascade argument does not apply. Although it is an easy matter to prove that the transformation A_0 is invertible, to prove that the whole transformation ϕ is invertible seems to be a difficult task and as yet the authors have not yet been able to develop such a proof. Until such a proof has been developed, whether or not the E-filter is homomorphic remains an open question.

Another unknown is what the generalized superposition operation is if the filter is homomorphic. The authors have spent very little time on this aspect of the theory as it seemed irrelevant to the particular application that they had in mind. A brief mention of a different superposition concept is made in Section III.

II. Realization of E-Filter in Real Time

At first thought, an E-filter might appear to be a rather impractical device when realized in the form shown in Fig. 2. We now proceed to show that, in general, the filter can be realized in real time. First of all we must obtain an analytic form for the overall transformation ϕ for the filter.

Now the output from the system A_0 is given by

$$f(e) = s[\theta^{-1}(e)].$$

This function is then operated on by a linear filter with causal impulse response $h(\cdot)$. The output of the linear filter will be

$$f^*(e) = \int_0^\infty f(e-u)h(u)\,du \tag{10}$$

i.e.,

$$f^*(e) = \int_0^\infty s[\theta^{-1}(e-u)]h(u)\,du.$$

$f^*(e)$ is then mapped back to the time domain by A_0^{-1} to give the output

$$r(t) = f^*[\theta(t)] \tag{11}$$

which becomes

$$r(t) = \int_0^\infty s[\theta^{-1}(\theta(t)-u)]h(u)\,du. \tag{12}$$

The preceding equation defines the transformation ϕ in terms of the time transformation θ and the linear filter response $h(\cdot)$.

In order to realize the filter in real time, we will need to manipulate (12) into a more convenient form. Let $e - u = \theta(t-\tau)$. Substituting, we obtain

$$r(t) = \int_0^\infty \dot\theta(t-\tau)s[\theta^{-1}[\theta(t-\tau)]h(u)]\,d\tau$$

where $u = \theta(t) - \theta(t-\tau)$, and so

$$r(t) = \int_0^\infty \dot\theta(t-\tau)s(t-\tau)h[\theta(t)-\theta(t-\tau)]\,d\tau$$

i.e.,

$$r(t) = \int_0^\infty \dot\theta(t-\tau)s(t-\tau)h\left[\int_{t-\tau}^{t}\dot\theta(t)\,dt\right]d\tau. \tag{13}$$

Equation (13) can be interpreted as follows. The input signal $s(t)$ is first operated on by multiplying it by $\dot\theta(t)$. The output of the multiplier then passes through a (time-varying) linear filter whose impulse response is dependent on $h(\cdot)$, (the impulse response of the time-invariant linear filter of Fig. 2) and the input $s(t)$ (via $\dot\theta(t)$) up to time t. See Fig. 3.

Provided the function $h(\cdot)$ is zero for negative values of its argument (i.e., $h(\cdot)$ is causal), it is easily seen that the system will work in real time. In the case where $h(\cdot)$ is noncausal, the filter can be realized in real time provided a suitable delay (related to the maximum rate of change expected in the input) is tolerated. A more detailed realization of a particular type of E-filter will be given in the next section.

III. An Example of E-Filter and Some of Its Properties

A linear filter can be qualitatively described as a system that presents different transfer characteristics to distinct classes of input signals. Hence a linear low-pass filter ideally would present a unity gain to "low-frequency signals" and zero gain to "high-frequency signals." If high-frequency and low-frequency signals are superimposed by addition, then the filter still treats them in the same way. Oppenheim's homomorphic filter exhibits similar behavior for signals that have been superimposed by operations other than addition. For example a "low-frequency signal" which has been multiplied by a "high-frequency signal" can be suitably filtered using homomorphic filtering.

A rather different but analogous problem arose in the authors' work on pattern recognition. The problem can be stated qualitatively as follows. Consider two signal classes S_1 and S_2 both occupying the same frequency band, but differing in that the signals in class S_1 have "low amplitudes" (noise signals) and the signals in S_2 have "high amplitudes." If a signal in S_1 is superimposed on a signal S_2 by addition, how can we filter out S_1 and leave S_2 relatively intact?

Consider for example the signal $s(t)$ shown in Fig. 7(b); this could be thought of as a large-amplitude signal with four large peaks (two "sharp" peaks and two "broad" peaks) to which has been added low-amplitude "noise." The low-amplitude noise can be removed by a *linear* low-pass filter, but such a filter will also remove the two narrow sharp peaks. What we require is a filter that does remove the noise but leaves the peaks relatively unaffected. Since a linear filter can't accomplish this task, we will need a nonlinear filter. We now describe such a filter, an E-filter, that accomplishes this task amazingly well.

Let the transformation A_0 be obtained by letting $f(e)$ be the value of $s(t)$ at a distance e along the graph of the function (see Fig. 4).

It is a simple matter to show that in this case (7) becomes

$$e = \int_0^t \sqrt{1 + (\dot{s}(t))^2}\, dt. \tag{14}$$

Let us assume that the linear filter L is a low-pass filter with a cutoff frequency f_c, and now let us consider the response of such an E-filter to a periodic signal of frequency f.

A. Response of Filter to Periodic Signal

Let $s(t)$ be a periodic signal with a frequency of repetition of f. Thus $s(t) = s(t + T)$, where $T = 1/f$. Let us transform the signal $s(t)$ to the E-domain by performing the transformation A_0, i.e.,

$$f(e) = A_0[s(t)]. \tag{15}$$

We will call $f(e)$ the E-transform of $s(t)$. Since $s(t)$ is periodic, $f(e)$ will be periodic. To find the period T_e of $f(e)$, we substitute $t = T$ in (14) to get

$$T_e = \int_0^T \sqrt{1 + (\dot{s}(t))^2}\, dt. \tag{16}$$

Now note that T_e satisfies the inequality $S_0(T) \le T_e \le S_1(T)$, where

$$S_0(T) = \int_0^T \sqrt{(\dot{s}(t))^2}\, dt = \int_0^T |\dot{s}(t)|\, dt \tag{17}$$

and

$$S_1(T) = \int_0^T 1 + \sqrt{(\dot{s}(t))^2}\, dt = T + S_0(T). \tag{18}$$

Thus if T is small the period of the E-transform of $s(t)$ approximately satisfies

$$T_e \doteq S_0(T) \tag{19}$$

with a maximum error of T.

But if $s(t)$ is symmetrical about the t axis and only has one maximum and one minimum per cycle (such as a square, triangular, sawtooth, trapezoid, or sinusoidal wave) then

$$S_0(T) \doteq 4s_{\max} \tag{20}$$

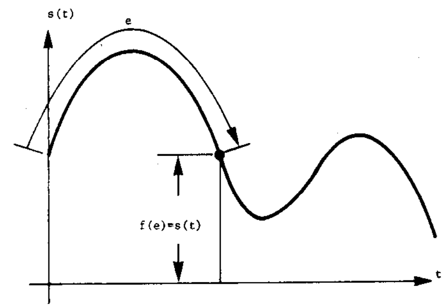

Fig. 4. Transformation to E-domain.

where $s_{\max}$ is the maximum amplitude of $s(t)$. Thus for the class of waveforms specified,

$$T_e \doteq 4s_{\max} \tag{21}$$

and so a periodic signal will pass through the filter provided it satisfies the condition

$$s_{\max} > \frac{1}{4f_c} \tag{22}$$

in the case of T small; i.e.,

$$T \ll 4s_{\max}. \tag{23}$$

We shall define the ratio of $s_{\max}$ and T as the *aspect ratio* A_r of the waveform, i.e.,

$$A_r \triangleq \frac{s_{\max}}{T}. \tag{24}$$

The property described in (22) states that for periodic waveforms, symmetric about the t axis, which have an aspect ratio A_r large with respect to 0.25, the E-filter processes signals in a way which is *independent of the signal frequency and purely depends on the signal amplitude.*

B. Details of E-Filter Realization

Fig. 5 is a schematic diagram of an E-filter of the type described in Section III-A. The time-invariant linear filter L in this case is a low-pass filter with a rectangular impulse response; i.e.,

$$h(e) = \begin{cases} 0, & e < 0 \\ 1, & 0 \le e \le a \\ 0, & a < e. \end{cases} \tag{25}$$

The schematic diagram is a more detailed representation of the general schematic in Fig. 3. It is made up of four basic units labelled A, B, C, and D. Unit A is the multiplier and B is the time-varying linear filter. Unit C provides the signal $\dot{\theta}(t)$ for the multiplier and unit D. Note that $\dot{\theta}(t)$ is always nonnegative. Unit D uses the signal $\dot{\theta}(t)$ to provide the correct control signals to the gates of B so that the impulse response of B depends on the signal $s(t)$ in the desired way.

At any particular time the first K of the gates in B will be turned on, allowing signals from the first K delay lines to be summed at the output. At this particular time, the time-varying filter has a rectangular impulse response of width $K\Delta$, where Δ is the incremental delay along the line.

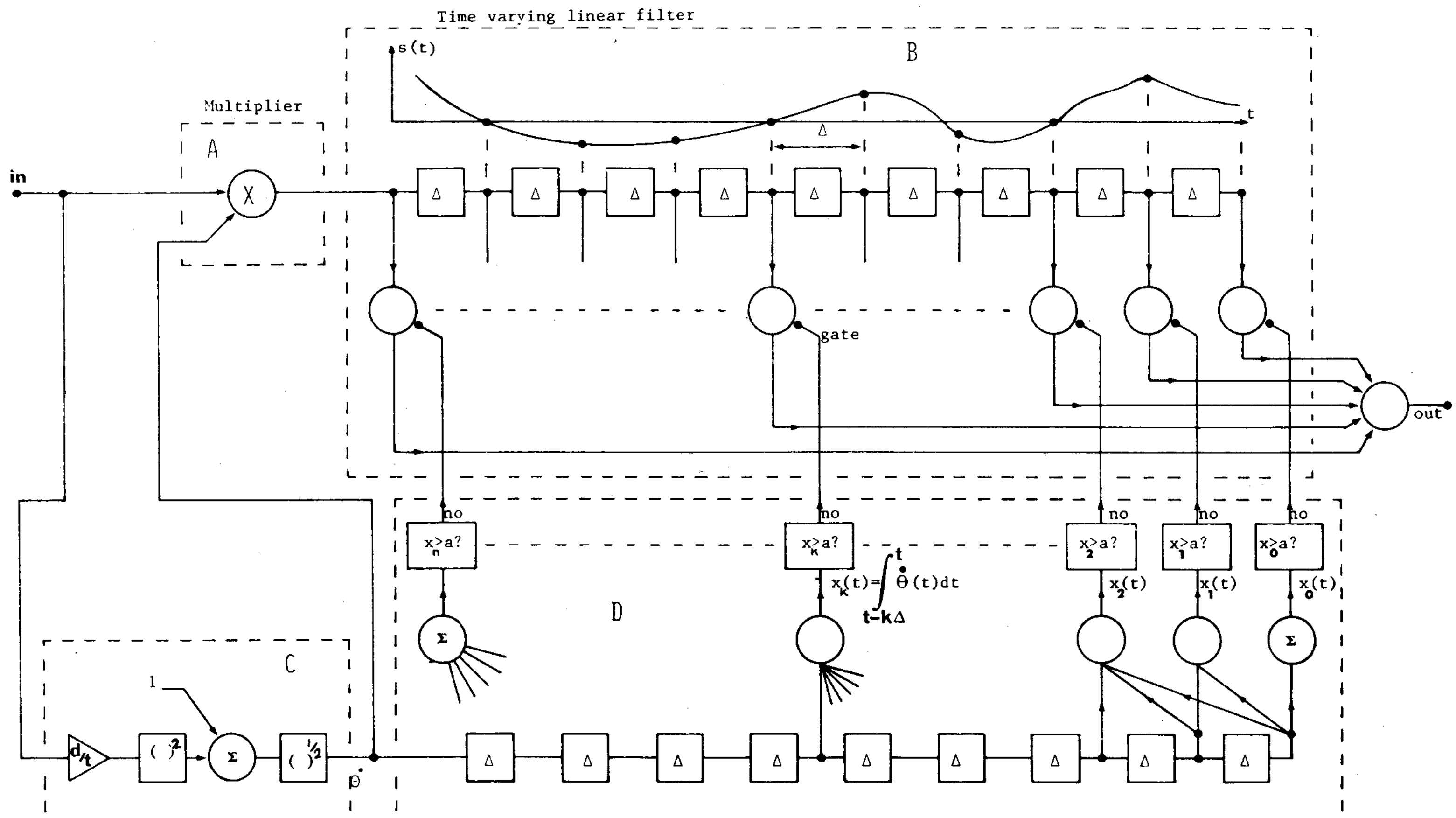

Fig. 5. Real-time realization of *E*-filter example.

The number K is determined at any time by the value of

$$\int_{t-\tau}^{t} \dot{\theta}(t)\, dt \tag{26}$$

at time t (refer to (13)). The function defined by (26) is calculated for discrete increments in τ at the outputs of the summing nodes in D.

C. Some Numerical Examples

Some numerical examples were obtained by a computer simulation of the nonreal-time E-filter algorithm. It was not until the completion of the examples that the real-time result occurred to us. We have recently repeated some of the examples using the real-time algorithm and obtained almost identical results. The derivative used was the second central difference.

The nonreal-time algorithm worked as follows. If the input signal is $S(K)$, $K = 1,2,\cdots,n$, then the signal in the E-domain is given by

$$f\left(\sum_{i=1}^{K} \sqrt{1 + (S(i) - S(i-1))^2}\right) = S(K), \qquad K = 1,2,\cdots,n.$$

In order to filter this signal we need apply some form of interpolation between the discrete points in the E-domain. We simply used linear interpolation.

Figs. 6 and 7 show plots of the input, the E-domain and output waveforms. The aspect ratio of the input signal has been set such that the inequality in (23) is satisfied. In Fig. 6 the response of the filter to various combinations of harmonically related sinusoids is shown. There is a slight distortion of the waveform which is particularly noticeable for the single sinusoid input. This distortion is due to the asymmetry of the impulse response $h(\cdot)$ of the linear low-pass filter L. If $h(\cdot)$ were allowed to be noncausal, this distortion could be overcome by using a symmetric function where

$$h(e) = h(-e).$$

If the symmetrical type of noncausal linear filter were used, then this would necessitate a suitable delay in the output response in order for the system to be realized in real time.

Fig. 7 shows a possible smoothing application of the filter, where the information of interest in the input function $s(t)$ is encoded in peaks of large amplitude but of variable "width" with superimposed noise. If a linear (low-pass) filter were used to eliminate the noise signal, then the sharp peaks would be severely attenuated with respect to the broad peaks.

In Section III-A we showed that the particular E-filter considered here has a response to periodic, high-aspect-ratio signals (i.e., signals that have been preamplified sufficiently) which is not dependent on frequency but merely amplitude. The low-amplitude signals are attenuated while the high-amplitude periodic signals pass through the filter. The examples indicate that this qualitative assessment of the filter performance also applies to the case of two superim-

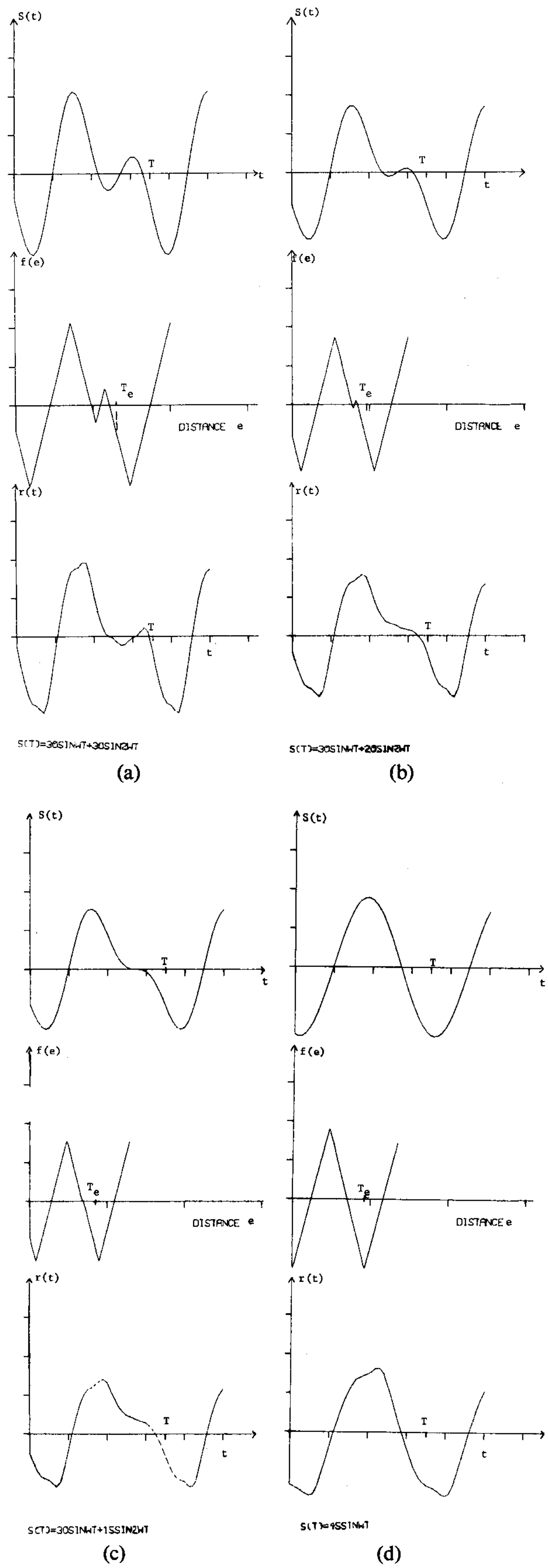

Fig. 6.

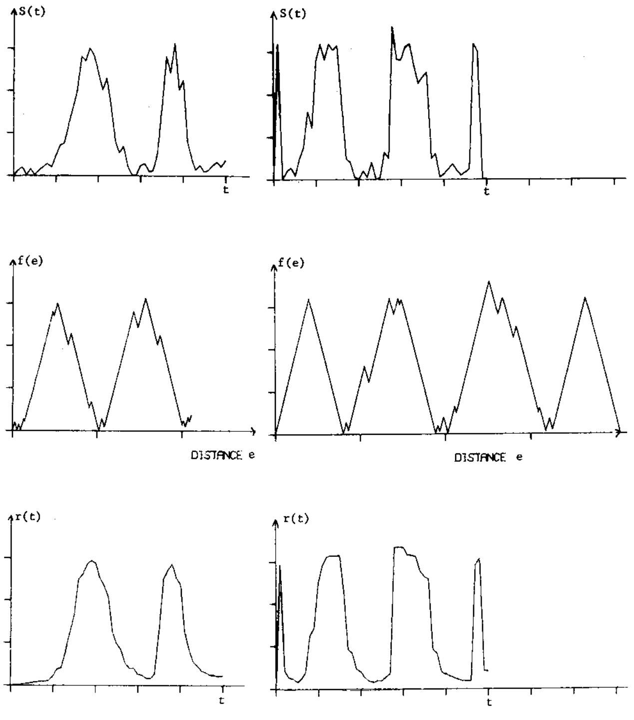

Fig. 7.

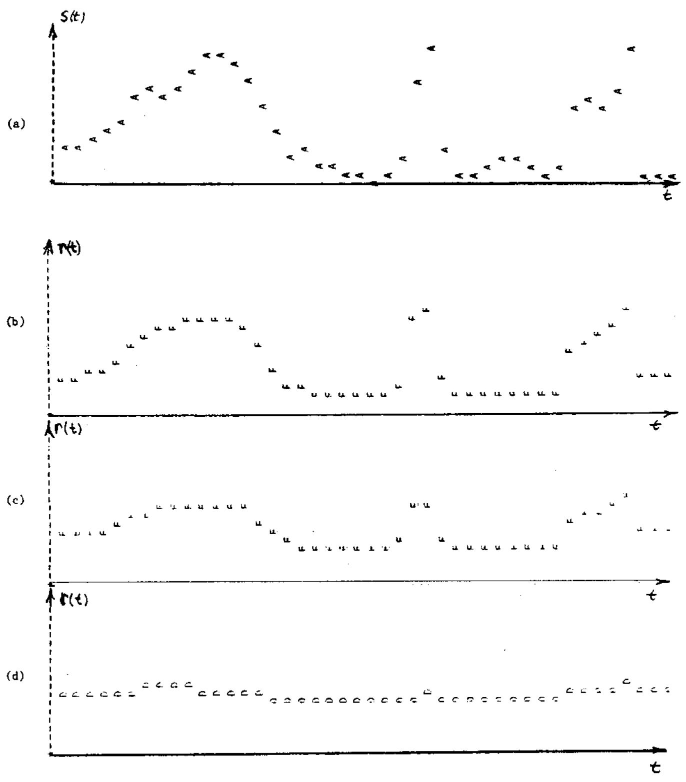

Fig. 8. (a) Input waveform. (b) Output waveform with high aspect ratio input. (c) With lower aspect ratio input. (d) With low aspect ratio input.

posed nonperiodic signals, i.e., to the situation of a large-amplitude signal of unspecified frequency with a superimposed high-frequency, low-amplitude noise signal. The noise component is filtered out leaving the large signal with its large peaks. As can be seen from Fig. 7, the amplitude of the narrow and the broad peaks remain unaffected by the filter.

In Fig. 8 a noncausal low-pass filter has been used for L and consists of a Hanning window. The output is shown for inputs of various aspect ratios, i.e., for decreasing input amplitude. It can be seen that for low-amplitude, low-frequency signals, the filter just acts as a low-pass filter with the impulse response of the Hanning window. When the signal is presented at a high amplitude, the E-filter becomes a time-varying low-pass filter and filters out the low-amplitude variations. Fig. 9 shows some further examples of inputs and outputs for this filter. Note that the filter seems to perform in a way which is independent of its dc level. This is shown in the next section to be a general characteristic of a large class of E-filters.

The application of the real-time result derived in this paper greatly decreases computer storage requirements when implementing the filtering as a computer algorithm. One particular point that must be considered when applying this result is the problem of the differentiator and the sampling rate of the input signal. This is basically a problem of discretization. If the sampling rate of the input signal is an order of magnitude higher than the Nyquist rate, no discretization problems should occur.

D. Further Property of E-Filter

The filtering action of the E-filter is independent of the dc level of the input signal if certain conditions are imposed on the θ transformation. This property is of practical importance, and so we shall formalize the statement in the form of a theorem.

Theorem: Let ϕ represent the filter transformation in the canonic form shown in Fig. 2, where $s(\cdot)$ and $r(\cdot)$ denoted the corresponding input and output waveforms.

Let the condition

$$\theta[s(v) + K, t] \triangleq \theta[s(v),t], \qquad 0 \le v \le t \tag{27}$$

be imposed on the θ transform. Then the filter input/output relationship satisfies the following. If

$$r(t) = \phi[s(t)]$$

then

$$r(t) + K = \phi[s(t) + K]. \tag{28}$$

Proof: The ϕ transformation was defined in (12) in terms of the time transformation θ and the linear filter impulse response $h(\cdot)$ as

$$r(t) = \int_0^\infty s(\theta^{-1}[\theta(t) - u])h(u)\, du \tag{29}$$

where

$$\theta = \theta[s(v),t], \qquad 0 \le v \le t.$$

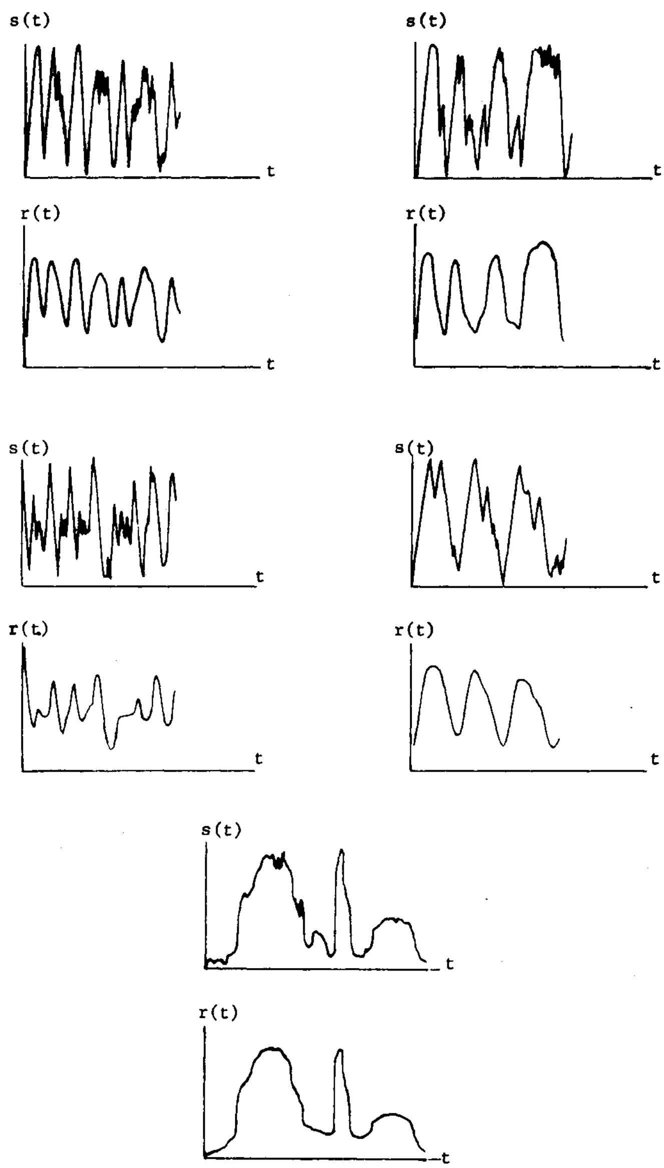

Fig. 9. Further input–output examples of low-pass E-filter.

If we make the substitution

$$s(t) = s_1(t) + K \tag{30}$$

in (29), then we need only show that

$$r(t) = r_1(t) + K$$

where

$$r_1(t) = \phi[s_1(t)] \tag{31}$$

to prove the theorem. From (29) and (30), we obtain

$$r(t) = \int_0^\infty \{s_1[\theta^{*-1}[\theta^*(t) - u]] + K\}h(u)\,du$$

where

$$\theta^* = \theta^*[s_1(v) + K, t], \qquad 0 \le v \le t.$$

By imposing the condition on the θ transformation expressed in (27), and knowing that $h(\cdot)$ is the causal impulse response of a linear filter, the equation can be simplified to

$$r(t) = \int_0^\infty s_1[\theta^{-1}[\theta(t) - u]]h(u)\,du + K.$$

We have also assumed that the dc gain is unity. This involves no loss of generality.

Using (30), this can be simplified to $r(t) = r_1(t) + K$ which completes the proof.

It is reasonably obvious that the transformation

$$\theta(s(v),t) = \int_0^t \sqrt{1 + [\dot{s}(v)]^2}\,dv$$

satisfies the condition expressed in (27). Consequently, the filter discussed earlier in this section has the property expressed in (28), i.e., its filtering action is independent of the dc level of the input signal.

IV. Conclusions

The results reported in this paper arise from a preliminary investigation of a new and novel form of nonlinear filter called the E-filter. It is shown that the filter can be realized in real time. When the real-time result is utilized in the form of an algorithm on a digital computer it results in a large economy in computer storage.

It is shown that for a wide class of E-filters, the "filtering" is independent of the dc level of the input signal. The filter is nevertheless amplitude dependent and can be designed to filter out low-amplitude signal excursions and leave the higher-amplitude excursions intact. Unlike a linear low-pass filter, the low-pass E-filter has a filtering action which is independent of the rise time of the peaks and more dependent on the amplitude of the input signal. In other words, the low-pass E-filter described in this paper filters out the superimposed "noise" on large peaks of the input signal.

The authors have found the low-pass E-filter of immense use in the filtering and peak extraction of the "pattern functions" that occur in their work in pattern recognition [9]–[14].

The E-filter should also find wide application in the processing of signals such as EEG, EMG, or voice signals, and anywhere where information is encoded in the amplitude or peaks of the signal.

Acknowledgment

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Invariant Gauss–Gauss Detection

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Abstract—**The detection of information-bearing Gaussian processes immersed in additive white Gaussian noise (WGN) is an important problem that arises in many signal processing applications. When the level of the WGN is unknown, classical approaches to the problem fail. In this paper a principle of invariance is used to derive a detector with performance that is invariant (or insensitive) to system gain, or equivalently channel attenuation. The detector structure can be realized and detection thresholds set without prior knowledge of the WGN level. When the observation interval is large the detector has the structure of a *spectral* estimator-correlator, the output of which is compared to an adaptive threshold. The invariance feature of the detector makes it a constant false alarm rate (CFAR) receiver; an ad hoc structure for suboptimal CFAR Gauss–Gauss detection is discussed as well.**

I. Introduction

A COMMON problem in passive signal detection is that of detecting a nonwhite Gaussian process immersed in additive white Gaussian noise (WGN). Typically, the nonwhite process is an information-bearing signal (as in passive sonar, radar, or radio astronomy) and the WGN is a model for broad-band acoustic or electromagnetic interference (noise). Most treatments of the Gauss–Gauss detection problem [1]–[7] have focused primarily on the structure of the detector without much discussion of the required statistical procedures to handle unknown parameters such as the unknown WGN level. A knowledge of the WGN level is assumed in order to configure classical detector structures and/or set decision thresholds. This is a severe limitation because in practical applications the WGN level is almost always unknown.

The detector structure discussed in this paper overcomes the requirement to know the WGN level and can, in fact, be implemented without such knowledge. The structure exhibits a very natural and desirable invariance (insensitivity) to system gain and channel attenuation. A consequence of this fact is that the detector is a constant false alarm rate (CFAR) receiver. The detector is shown to be approximately optimum-invariant when the signal-to-noise ratio is small. Since any CFAR detector for the Gauss–Gauss problem must be invariant to system gain, the invariant detector is, within the limits of the approximation, the best achievable CFAR detector.

The spectral estimator-correlator part of the invariant detector is identical to Price's "practically optimum" processor [8]. However, Price's detector requires prior knowledge, or measurement during a signal-free observation interval, of the WGN level in order to set detection thresholds. The normalizing feature of our detector results in an invariance property that, in principle, permits setting of detection thresholds even when the WGN level is unknown.

The statistic required to implement the invariant detector has a distribution which in general cannot be obtained in closed form. However, the statistic is similar to quadratic forms that have been studied extensively in the statistical literature [19], [20]. Several methods of approximating densities for these quadratic forms are discussed.

The form of the (approximately) optimum detector suggests an ad hoc structure that may be used for suboptimal detection in problems where the signal process is very narrowband with respect to the noise process. Several remarks are advanced concerning the distribution of the suboptimal detector. In fact, an example is worked for which the distribution of the detector can be obtained in closed form.

Several authors, including Siebert [9], Spooner [10], Picinbono and Vezzoni [11], [12], and Scharf and Lytle [13], have treated the problem of detecting known-form signals in Gaussian noise of unknown level. Tuteur [14] has treated the problem of detecting transiting noise sources

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